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# Note

# On the Spectrum of $n$ Quasigroups with Given Conjugate Invariant Subgroup

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We show in this paper that for any group $G$ of permutations of an $(n+1)$ set, any composite $q \geqslant (n+1)^2$ is the order of an $n$ quasigroup $Q$ whose conjugate invariant subgroup is precisely $G$.

If $n$ is a positive integer, and $X$ is a finite set of cardinality $|X| = q$, we say that $Q$ is an *$n$ quasigroup* on $X$ of *order* $q$ if $Q$ is a $(q^n) \times (n+1)$ array, with entries from $X$, with the property that each of the $n+1$ $q^n \times n$ subarrays of $Q$ consists of $q^n$ distinct rows. Thus each of the $q^n$ possible $n$ tuples with entries from $X$ occurs exactly once as a row in such a subarray. We consider two $n$ quasigroups on $X$ *equivalent* if a permutation of the rows of one produces the other.

If $Q$ is an $n$ quasigroup on $X$, we denote by $\Gamma(Q)$ the set of permutations of the columns of $Q$ which produce an array equivalent to $Q$. Evidently, $\Gamma(Q)$ is a subgroup of the symmetric group $S$ of permutations of $\{0, 1,..., n\}$. We call $\Gamma(Q)$ the *conjugate invariant subgroup* of $Q$. These have been studied previously (see, for example, [6, 7]).

For each subgroup $G$ of $S$, we define $\Lambda(G)$, the *spectrum* of $G$, to be the set of all positive integers $q$ for which there exists an $n$ quasigroup $Q$ of order $q$ with $\Gamma(Q) = G$.

We are concerned here with the following:

*Conjecture.* For each subgroup $G$ of $S$, $\Lambda(G)$ consists of all but finitely many positive integers.

If $\Gamma(Q) = S$, the $n$ quasigroup $Q$ is said to be *totally symmetric*. These exist for all orders; for example, take for the rows of $Q$ the $(n+1)$ tuples summing to zero in an Abelian group $X$. Thus $\Lambda(S)$ contains all positive integers, so the conjecture is true if $G = S$.

Lindner and Steedly [2] have established the conjecture for $n = 2$, i.e.,



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$q^2 \times 3$ orthogonal arrays. McLeish has established the conjecture for $n = 3$ in most cases in [3–5], as well as the case $|G| = 1$ for all $n$. We showed in [1] that $\Lambda(G)$ is infinite if $G$ is a semiregular group of permutations.

Our main result is:

THEOREM. *Let $m$ and $p$ be positive integers with $m > n$ and $p \geqslant 2$. Then for any subgroup $G$ of $S$, there is an $n$ quasigroup $Q$ of order $mp$ with $\Gamma(Q) = G$. In particular, every composite $q \geqslant (n+1)^2$ is in $\Lambda(G)$.*

*Proof.* Let $M$ and $P$ be finite Abelian groups of orders $m$ and $p$, respectively. We construct an $n$ quasigroup $Q$ on $X = M \times P$ as follows: Choose $n+1$ distinct elements $a_0, a_1, ..., a_n$ in $M$, choose a pair $b$, $c$ of distinct elements of $P$, and let $s = a_0 + a_1 + \cdots + a_n$.

Now let $x_0, x_1, ..., x_n \in M$, $y_0, y_1, ..., y_n \in P$. We declare the $(n+1)$ tuple $((x_0, y_0), (x_1, y_1), ..., (x_n, y_n))$ to be a row of $Q$ if and only if the following two conditions hold:

(1) $x_0 + x_1 + \cdots + x_n = s$;

(2) if for some $\pi \in G$, $(x_0, x_1, ..., x_n) = (a_{\pi(0)}, a_{\pi(1)}, ..., a_{\pi(n)})$, then $y_0 + y_1 + \cdots + y_n = b$. Otherwise, $y_0 + y_1 + \cdots + y_n = c$.

It is easy to check that $Q$ is an $n$ quasigroup on $X$, and that $\Gamma(Q) = G$.

Finally, if $q \geqslant (n+1)^2$ is composite, then $q$ has a prime factor $p \leqslant \sqrt{q}$, and then $m = q/p > n$.

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